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# DYNAMIC CHARACTERISTICS OF A VARIABLE-MASS ELASTIC BODY UNDER HIGH ACCELERATIONS

Semi-Annual Technical Progress Report

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ABSTRACT

The dynamic characteristics of a variable-mass elastic system are under investigation. The mathematical model consists of a slender, elastic case, closed at one end and open at the other end, with internal gas flow. The model represents a solid-fuel missile.

In the preceding six-month period the problem has been formulated and the general equations of motion derived. In the six-month period covered by this report particular attention has been given to the internal gas flow problem. Furthermore, the vertical flight case has been used to study the meaning of normal mode vibration. It appears that for a variable-mass systems one cannot speak of natural frequencies and normal modes, in an ordinary sense, although a solution in terms of the eigenfunctions of the corresponding constant-mass system is assumed. This solution should be regarded as a mathematical convenience with no particular physical significance attached.

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## 1. Introduction

The present semi-annual technical progress report covers the period December 1, 1965 - May 31, 1966. The first semi-annual technical report covering the preceding six-month period contains the problem description and formulation. This will not be repeated here and we shall extract the pertinent equations from the first report.

The motion of a missile, envisioned as a slender elastic body of variable-mass, is described by means of three rigid-body coordinates, $X(t)$, $Y(t)$ and $\theta(t)$, and two elastic-body displacements, $u(x,t)$ in the axial direction and $y(x,t)$ in the transverse direction. From section 7 of the first semi-annual report we extract the equations of motion, Eqs. (7.29) through (7.33), and the associated boundary conditions, Eqs. (7.34) through (7.36). The equations of motion are

$$- M_c(\ddot{X} - \ddot{\theta}Y - 2\dot{\theta}\dot{Y} - \dot{\theta}^2 X) - \int_0^L m_c \ddot{u}\, dx + \ddot{\theta}\int_0^L m_c y\, dx + \dot{\theta}^2 \int_0^L m_c(x + u)\, dx$$

$$+ 2\dot{\theta}\int_0^L m_c \dot{y}\, dx - \int_0^L (\bar{f}_{RF} \cdot \bar{i} + \dot{m}_c v)\, dx - M_c g \sin\theta + F_{Xc} = 0 \qquad (1.1)$$

$$- M_c(\ddot{Y} + 2\dot{\theta}\dot{X} + \ddot{\theta}X - \dot{\theta}^2 Y) - \int_0^L m_c \ddot{y}\, dx - \ddot{\theta}\int_0^L m_c(x + u)\, dx + \dot{\theta}^2 \int_0^L m_c y\, dx$$

$$- 2\dot{\theta}\int_0^L m_c \dot{u}\, dx - \int_0^L (\bar{f}_{RF} \cdot \bar{j})\, dx - M_c g \cos\theta + F_{Yc} = 0 \qquad (1.2)$$

$$- (\ddot{\theta}Y + \dot{\theta}^2 X - \ddot{X} + 2\dot{\theta}\dot{Y})\int_0^L m_c y\, dx + (-\ddot{\theta}X + \dot{\theta}^2 Y - \ddot{Y} - 2\dot{\theta}\dot{X})\int_0^L m_c(x + u)\, dx$$

$$- \int_0^L m_c[\ddot{y}(x + u) - \ddot{u}\, y]\, dx - \ddot{\theta}\int_0^L m_c[(x + u)^2 + y^2]\, dx$$

$$- 2\dot{\theta}\int_0^L m_c[(x + u)\dot{u} + y\dot{y}]\, dx + \int_0^L [(\dot{m}_c v + \bar{f}_{RF} \cdot \bar{i})y - (\bar{f}_{RF} \cdot \bar{j})(x + u)]\, dx$$

$$-\int_0^L m_c g[(x + u)\cos\theta - y\sin\theta]\,dx + YF_{Xc} - XF_{Yc} + F_{\theta c} = 0 \tag{1.3}$$

$$\frac{\partial}{\partial x}\left(EA_c \frac{\partial u}{\partial x}\right) - m_c[\ddot{X} + \ddot{u} - \ddot{\theta}(Y + y) - 2\dot{\theta}(\dot{Y} + \dot{y}) - \dot{\theta}^2(X + x + u)]$$
$$- \dot{m}_c v - \bar{f}_{RF} \cdot \bar{i} - m_c g \sin\theta + \hat{F}_{uc} = 0 \tag{1.4}$$

$$-\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) + \frac{\partial}{\partial x}\left(P \frac{\partial y}{\partial x}\right) - m_c[\ddot{Y} + \ddot{y} + \ddot{\theta}(X + x + u) + 2\dot{\theta}(\dot{X} + \dot{u})$$
$$- \dot{\theta}^2(Y + y)] - \bar{f}_{RF} \cdot \bar{j} - m_c g \cos\theta + \hat{F}_{yc} = 0 \tag{1.5}$$

where the latter two equations are subject to the boundary conditions

$$EA_c \frac{\partial u}{\partial x} = 0 \qquad \text{at } x = 0,\ L \tag{1.6}$$

$$\left.\begin{aligned} EI \frac{\partial^2 y}{\partial x^2} &= 0 \qquad \text{at } x = 0,\ L \\ -P \frac{\partial y}{\partial x} + \frac{\partial}{\partial x}\left(EI \frac{\partial^2 y}{\partial x^2}\right) &= 0 \qquad \text{at } x = 0,\ L \end{aligned}\right. \tag{1.7}$$

The equations of motion and the boundary conditions involve such quantities, some of them in an implicit way, as the stiffness of the shell, the missile mass distribution at any time, drag forces as well as internal viscous effects, mass flow rates, internal pressure and fluid velocity distribution, etc. It follows that to be able to investigate the dynamic characteristics of the vehicle one must have additional relations at his disposal. The internal flow problem of a missile is a relatively complex problem in itself even for the case in which the missile is stationary. An approximation of constant chamber pressure is often used, together with the resulting linear velocity distribution of the gas flow. This assumption ignores the pressure drop due to flow acceleration as well as the effects of internal friction and, in addition, the fact that the control volume is accelerating. The assumptions of constant pressure and linear velocity will be examined. The terms resulting from the internal gas flow which appear in the equations of motion can be regarded as forcing functions.

We shall make the assumption that initially the mass distribution is uniform and, furthermore that rate of fuel burning is the same for any point in the missile which implies that the burning rate is independent of pressure and temperature. As a result, the mass distribution does not depend on the spatial coordinate x but only on the time t

$$m_c\,(x,t) = m_c(t) \tag{1.8}$$

This also implies that the center of mass of the missile does not shift with respect to the missile. Since the center of mass is stationary relative to the missile, at the half-way point between the ends of the missile, we shall find it convenient to measure x from the center of the missile instead of the aft end. This fact becomes immediately evident since in this case we have $\int_{-L/2}^{L/2} m_c\, x\, dx = 0$. One should also note that, in the process, X(t) and Y(t) become the coordinates of the center of mass.

The equations of motion, Eqs. (1.1) through (1.5) are highly nonlinear and coupled. To add to complexity the mass distribution is time-dependent. The closed form solution of the equations in the present form is not possible. The purpose of this research is to study the dynamic characteristics of the missile so that an analysis, even under simplifying assumptions, is highly desirable. An entirely numerical solution of the equations would tend to obscure these characteristics but, nevertheless, will be performed to check the validity of the assumptions.

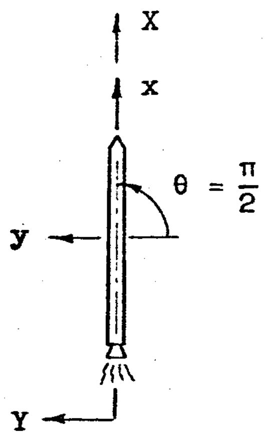


Figure 1

The flight of many sounding rockets consists of a vertical, upright flight. We shall be interested in studying this case in more detail and, to this end, we wish to define a "primary motion" as the vertical rigid-body flight of the vehicle. The deviations from the primary motion will form a "secondary motion" and will be regarded as small perturbations. Finally, the equations of motion are linearized by assuming small elastic displacements.

Although in their final form the equations of motion are linear, their solution is by no means an ordinary task. It should be recalled that the mass distribution is time-dependent and the vast majority of literature treating the subject of elastic missiles is concerned with constant-mass systems. For variable-mass systems, the normal modes, in the commonly accepted sense, for the elastic motion have no physical meaning. Nevertheless, one still can use a series solution using the eigenfunctions of a related constant-mass system for the purpose of eliminating the spatial variable x from the partial differential equations for the elastic axial and transverse motions. Two sets of ordinary differential equations with time-dependent coefficients, one for the axial motion and one for the transverse motion, are obtained. Fortunately, due to the uniform-mass assumption, the set of equations for the axial motion is uncoupled and a solution can be obtained without much difficulty. On the other hand, the set of equations for the transverse motion is coupled and, in addition, some of the coefficients consist of axial motion terms. A closed form solution of such a set is not possible. One can obtain a solution, however, by retaining only the axial term corresponding to the first eigenfunction and the transverse terms corresponding to the first two eigenfunctions since, for such a case, the set of equations for the transverse motion consists of two uncoupled equations. This solution will be checked against an entirely numerical solution of the problem to verify to what extent one can justify using such a small number of terms to describe the axial and transverse motions.

It should be pointed out that, although some vibration is obtained, this is not a normal mode vibration. The system has no natural frequencies in the ordinary sense. The fact that we use eigenfunctions and eigenvalues corresponding to the related constant-mass system to expedite the solution should be regarded as a mathematical device with no particular physical significance attached.

## 2. The Internal Flow Problem.

From the point of view of fluid flow problem, the missile will be regarded as a cylindrical container of uniform cross-section, open through a nozzle at the end $x = -L/2$ and closed at the end $x = L/2$. We shall assume that the nozzle length is relatively short compared with the length of the missile. Since the end $x = L/2$ is closed, the burned fuel will flow in the negative x-direction so that at any point x the flow rate will be equal to the total rate of change of mass between the point x and the closed end. Denoting this quantity by $\dot{M}_c(x,t)$ (note that according to the previously used notation $\dot{M}_c(-L/2,t) = \dot{M}_c$) and using the continuity equation for the fluid, we have

$$\dot{M}_c(x,t) = \int_x^{L/2} \dot{m}_c(\xi,t)\, d\xi = -\rho_F A_F v(x,t) \tag{2.1}$$

where $\xi$ is a dummy variable of integration, $\rho_F$ is the fluid density and $A_F$ the fluid cross-sectional area. The negative sign accounts for the fact that the flow is in the negative x-direction. As a first approximation, one may assume that the pressure in the burning chamber is uniform (Ref. 4, p.42) and that the flow is frictionless. Hence, $\rho_F$ will be constant and, assuming that $A_F$ is constant, we have $m_F = \rho_F A_F = \text{const}$. Using Eq. (1.8), we obtain

$$v(x,t) = -\frac{\dot{M}_c(x,t)}{m_F} = -\frac{\dot{m}_c}{m_F}(L/2 - x) \tag{2.2}$$

so that $v(x,t)$ is a linear function of x.

One must also make some assumptions as to the mass flow rate as a function of time. This will be largely determined by the grain configuration, in addition to inhibitors, pressure transients, etc. A simple cylindrical grain shape results in approximately "neutral burning" whereas various cruciform or multiple grain forms may be designed to give either "progressive" or "regressive" burning. Typical mass flow rates amenable to analytic treatment may be approximated as shown by a rectangular function, or slightly more accurately by a trapezoidal function (see Ref. 1)

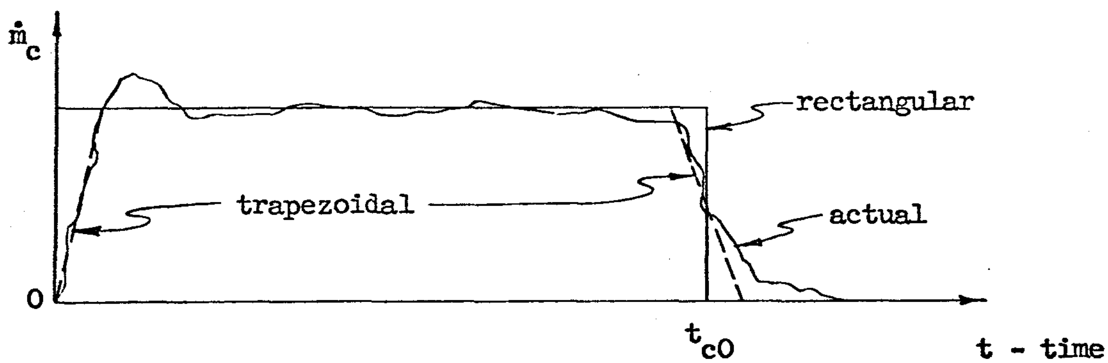



Figure 2

Let us for the moment relax the uniform pressure assumption and derive an expression of the pressure as a function of the spatial coordinate x. To this end we use the equation of motion for the fluid element (see Eq. (5.2) of the first semi-annual report), which for the axial fluid flow and planar missile motion reduces to

$$\bar{f}_{EXF} + \bar{f}_{INF} + \bar{f}_{RF} = m_F \ddot{\bar{R}}_c - 2\, m_F \dot{\theta} v \bar{j} + \left[ \frac{\partial}{\partial x} (m_F\, v^2) + \frac{\partial}{\partial t} (m_F\, v) \right] \bar{i} \tag{2.3}$$

Concentrating on the axial component and assuming steady-state, frictionless, constant-area flow, Eq. (2.3) yields

$$- \frac{\partial p}{\partial x} = \frac{\partial}{\partial x} (\rho_F\, v^2) + \rho_F\, \ddot{\bar{R}}_c \cdot \bar{i} \tag{2.4}$$

Assuming that $\ddot{\bar{R}}_c$ is known, we have one equation and three unknowns, p, $\rho_F$ and v. An additional equation is Eq. (2.1) where $\dot{M}_c(x,t)$ should be regarded as known, as is $A_F$ if the burning rate of the propellant is known (assumed independent of pressure and time). A third equation can be obtained by assuming a perfect gas

$$p = \rho_F \, R \, T \tag{2.5}$$

which introduces the absolute temperature T as a new unknown, since the gas constant R is assumed as known. Consistent with the perfect gas assumptions, we have the following relations

$$C_P - C_v = R, \qquad C_v = \frac{C_P}{\gamma} \tag{2.6}$$

where $C_P$ and $C_v$ are the specific heats and $\gamma$ is the specific heats ratio. The above equations can be supplemented by the energy-heat relations

$$h + \frac{1}{2} v^2 = h_{L/2}, \quad h - h_r = C_P \, T \tag{2.7}$$

where h is the enthalpy at any point and $h_{L/2}$ is the enthalpy at the closed end of the missile. The reference enthalpy, $h_r$, can be taken zero. Integrating Eq. (2.4) and using Eqs. (2.1) and (2.5), we obtain

$$p = p_{L/2} - \frac{p}{RT} v^2 + \int_x^{L/2} \frac{p}{RT} \ddot{\bar{R}}_c \cdot \bar{i} \, d\xi$$

$$= p_{L/2} + \frac{\dot{M}_c(x,t) \; v}{A_F} - \int_x^{L/2} \frac{\dot{M}_c(x,t)}{A_F v} \ddot{\bar{R}}_c \cdot \bar{i} \, d\xi \tag{2.8}$$

In addition, Eqs. (2.1), (2.5) and (2.7) yield

$$v^2 - \frac{2p\ A_F\ C_p}{R\ \dot{M}_c(x,t)}\ v - 2\ C_p\ T_{L/2} = 0 \tag{2.9}$$

which is a quadratic equation in v. One can solve Eq. (2.9) for v and introduce into Eq. (2.8) to eliminate its dependence on v. In most cases the contribution of the integral in Eq. (2.8) is relatively small and therefore neglected. This is equivalent to assuming an unaccelerated control volume. Ignoring the integral in Eq. (2.8) and introducing Eqs. (2.6), as well as the expression of v from Eq. (2.9), into Eq. (2.8) we obtain

$$\frac{p}{p_{L/2}} = \frac{1}{\gamma + 1} \pm \frac{1}{\gamma + 1}\left[1 + (\gamma - 1)(\gamma + 1)(1 - 2\ C_p\ T_{L/2}\ \frac{{A_F}^2}{\dot{M}_c{}^2(x,t)})\right]^{1/2} \tag{2.10}$$

which agrees with Price (see Ref. 3).

Next let us examine the relative magnitude of the integral in (2.8). For a typical high-acceleration missile with a maximum Mach number in the combustion chamber of 0.4 we have $p \cong 0.95\ p_{L/2}$, or a 5% pressure loss due to the flow acceleration. For $p_{L/2}$ = 2000 psi the pressure drop amounts to $\Delta p$ = 100 psi. To determine the magnitude of the noninertial term, we need $\ddot{\bar{R}}_c \cdot \bar{i}$. For no rotation, $\ddot{\theta} = \dot{\theta} = 0$, we have $\ddot{\bar{R}}_c \cdot \bar{i} = \ddot{X} + \ddot{u}$ and since $\ddot{u}$ is sign-variable we can ignore it. For a 10 in.-diameter missile of approximately 100 in. length, 425 lb. weight and subjected to $p_{L/2}$ = 2000 psi, the linear acceleration $\ddot{X}$ will be of the order of magnitude of 5000 ft sec$^{-2}$, depending on the nozzle configuration, drag, etc. Assuming the fluid density $\rho_F$ to be essentially uniform, the noninertial term produces a pressure change

$$\Delta p \cong \int_{-L/2}^{L/2} \rho_F\ \ddot{X}\ dx = 9\ \text{psi} \tag{2.11}$$

which is about 0.5% of the total pressure, hence, insignificant. Note that this is a pressure increase rather than a pressure drop so that it tends to render the pressure more uniform.

The magnitude of the friction term $\overline{f}_{EXF}$ may be estimated by assuming a value for the friction factor, $f = 0.005$ (see Wimpress, p. 33) and again assuming uniform $\rho_F$. Denoting by z the perimeter of the flow area the pressure drop due to friction can be written as

$$\Delta p = \int_{-L/2}^{L/2} \frac{f\, v^2\, z\, \rho_F}{2A_F}\, dx \cong 3.3 \text{ psi} \tag{2.12}$$

which is again very small.

Hence, the uniform pressure and linear velocity assumptions appear to be fairly reasonable.

3. The Vertical, Upright Flight.

We shall assume that the vertical, upright flight (see Figure 1) consists of a primary motion defined by the rigid-body vertical flight and a secondary motion defined as a perturbation about the primary motion. With this in mind, one can simplify Eqs. (1.1) through (1.3) by assuming $X >> Y$, $X >> \theta$ and $\theta \cong \frac{\pi}{2}$, so that second and higher order terms in Y and $\theta$ are ignored. These assumptions lead to the equations

$$- M_c\ddot{X} - \int_{-L/2}^{L/2} m_c\ddot{u}\, dx - \int_{-L/2}^{L/2} (\overline{f}_{RF} \cdot \overline{i} + \dot{m}_c v)\, dx - M_c g + F_{Xc} = 0 \tag{3.1}$$

$$- M_c(\ddot{Y} + 2\dot{\theta}\dot{X} + \ddot{\theta}X) - \int_{-L/2}^{L/2} m_c\ddot{y}\, dx - \ddot{\theta}\int_{-L/2}^{L/2} m_c u\, dx - 2\dot{\theta}\int_{-L/2}^{L/2} m_c\dot{u}\, dx$$
$$- \int_{-L/2}^{L/2} \overline{f}_{RF} \cdot \overline{j}\, dx + F_{Yc} = 0 \tag{3.2}$$

$$\ddot{X}\int_{-L/2}^{L/2} m_c y\, dx - (\ddot{\theta}X + \ddot{Y} + 2\dot{\theta}\dot{X})\int_{-L/2}^{L/2} m_c u\, dx - \int_{-L/2}^{L/2} m_c[\ddot{y}(x + u) - y\ddot{u}]\, dx$$

$$- \ddot{\theta} \int_{-L/2}^{L/2} m_c[(x + u)^2 + y^2]\, dx - 2\dot{\theta} \int_{-L/2}^{L/2} m_c[(x + u)\, \dot{u} + y\dot{y}]\, dx$$

$$+ \int_{-L/2}^{L/2} [(\bar{f}_{RF} \cdot \bar{i} + \dot{m}_c v)y - \bar{f}_{RF} \cdot \bar{j}\ (x + u)]\, dx + \int_{-L/2}^{L/2} m_c g\, y\, dx$$

$$+ YF_{Xc} - XF_{Yc} + F_{\theta c} = 0 \tag{3.3}$$

Equations (3.1) through (3.3) can be further simplified by assuming that the average elastic motion (even when multiplied by a weighting function) is zero and that the elastic displacements are sufficiently small so that second order terms can be neglected. With these assumptions, we obtain

$$\begin{aligned} &- M_c\ddot{X} - \int_{-L/2}^{L/2} (\bar{f}_{RF} \cdot \bar{i} + \dot{m}_c v)\, dx - M_c g + F_{Xc} = 0 \\ &- M_c(\ddot{Y} + 2\dot{\theta}\dot{X} + \ddot{\theta}X) - \int_{-L/2}^{L/2} \bar{f}_{RF} \cdot \bar{j}\, dx + F_{Yc} = 0 \\ &- \ddot{\theta} \int_{-L/2}^{L/2} m_c x^2\, dx - \int_{-L/2}^{L/2} (\bar{f}_{RF} \cdot \bar{j})\, x\, dx + YF_{Xc} - XF_{Yc} + F_{\theta c} = 0 \end{aligned} \tag{3.4}$$

which establishes the rigid-body nature of X, Y and θ.

For no engine gimbal angle, we have

$$\int_{-L/2}^{L/2} \bar{f}_{RF} \cdot \bar{j}\, dx = 0 \tag{3.5}$$

which implies that there is no transverse component due to fluid flow.*
Furthermore, for the study of the dynamic characteristics we shall concentrate on the equivalent of the free-vibration case and ignore the external forces

$$F_{Xc} = F_{Yc} = F_{\theta c} = \hat{F}_{uc} = \hat{F}_{yc} = 0 \tag{3.6}$$

*The Coriolis effect is accounted separately.

so that the last two of Eqs. (3.4) are identically satisfied by

$$Y = \theta = 0 \tag{3.7}$$

whereas the first one yieldsthe axial acceleration

$$\ddot{X} = - g - \frac{1}{M_c} \int_{-L/2}^{L/2} (\bar{f}_{RF} \cdot \bar{i} + \dot{m}_c v) \, dx \tag{3.8}$$

4. <u>The Elastic Motion</u>.

In view of the results and assumptions of the preceding section the equations for the elastic motion, Eqs. (1.4) and (1.5) become

$$\frac{\partial}{\partial x}\left(EA_c \frac{\partial u}{\partial x}\right) + m_c \ddot{u} = - m_c \ddot{X} - \bar{f}_{RF} \cdot \bar{i} - \dot{m}_c v - m_c g = f(x,t) \tag{4.1}$$

and

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(P \frac{\partial y}{\partial x}\right) + m_c \ddot{y} = 0 \tag{4.2}$$

where the axial force, P, is given by

$$P = EA_c \frac{\partial u}{\partial x} \tag{4.3}$$

Equation (4.1) is subject to boundary conditions (1.6) and Eq. (4.2) to boundary conditions (1.7).

It must be noted, at this point, that the differential equation and the boundary conditions for the axial elastic displacement $u(x,t)$ do not contain the transverse elastic displacement $y(x,t)$. On the other hand, the differential equation and the boundary conditions for $y(x,t)$ do contain $u(x,t)$, as can be seen from the expression for P. Hence, one must first solve for $u(x,t)$ and then for $y(x,t)$.

Let us assume the solution of Eq. (4.1) in the form of the series

$$u(x,t) = \sum_{i=1}^{n} \varphi_i (x) \, q_i (t) \tag{4.4}$$

where $q_i$ (t) are time-dependent generalized coordinates and $\varphi_i$ (x) are the solutions of the eigenvalue problem consisting of the differential equation

$$-\frac{d}{dx}\left(EA_c \frac{d\varphi}{dx}\right) = m_o \omega^2 \varphi \tag{4.5}$$

and the boundary conditions

$$EA_c \frac{d\varphi}{dx} = 0, \quad x = -L/2, L/2 \tag{4.6}$$

This eigenvalue problem corresponds to the axial vibration problem of a rod of mass per unit length $m_o$. The functions $\varphi_i$ are such that $\int_{-L/2}^{L/2} m_o \varphi_i \varphi_j \, dx = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta, and, in addition $\int_{-L/2}^{L/2} m_o \varphi_i \, dx = 0$. The latter expression is consistent with the zero average elastic motion postulated in the preceding section. In this case $m_o$ is taken as the initial mass per unit length of missile, hence independent of x. From the uniform burning assumed in Section 2 we conclude that the distributed mass at any time is

$$m_c(t) = m_o - \dot{m}_c t = m_o (1 - \beta t) \tag{4.7}$$

where $\beta = \dot{m}_c/m_o$. Note that in the derivations of Section 2 of the first semi-annual report the sign of the mass increment was such that $\dot{m}_c$ must be regarded as positive although the system loses mass.

Introducing Eqs. (4.4) and (4.7) into Eq. (4.1), and recalling that the functions $\varphi_i$ satisfy Eq. (4.5), we obtain

$$\sum_{i=1}^{n} m_o [(1 - \beta t) \ddot{q}_i + \omega_i^2 q_i] \varphi_i = f(x,t) \tag{4.8}$$

Multiplying Eq. (4.8) by $\varphi_j$ and integrating over the length of the missile we obtain a set of uncoupled ordinary differential equations

$$(1-\beta t)\,\ddot{q}_i + \omega_i^2\, q_i = U_i(t), \quad (i=1,2,---,n) \tag{4.9}$$

where

$$U_i(t) = \int_{-L/2}^{L/2} f(x,t)\,\varphi_i(x)\,dx, \quad (i=1,2,---,n) \tag{4.10}$$

play the role of generalized forces and contain terms due to vehicle acceleration and internal gas flow. Due to the relatively rapid transition to a steady-state situation consisting of constant burning (see Figure 2), the time dependency of $U_i(t)$ may be assumed in the form of a step function for the duration of the powered flight.

By making the substitution $1-\beta t = \tau^2$ in the homogeneous part of Eq. (4.9), we obtain

$$\tau \frac{d^2 q_i}{d\tau^2} - \frac{dq_i}{d\tau} + \lambda_i^2\, \tau q_i = 0, \qquad \lambda_i = \frac{2\omega_i}{\beta} \tag{4.11}$$

which has the solution

$$q_i(\tau) = C_{1i}\,\tau\, J_1(\lambda_i \tau) + C_{2i}\,\tau\, Y_1(\lambda_i \tau) \tag{4.12}$$

where $J_1$ and $Y_1$ are Bessel functions of the first order and first and second kind, respectively. The solution of Eq. (4.9) can be written in the form

$$q_i(t) = (1-\beta t)^{1/2}\left[C_{1i}\, J_1(\lambda_i\sqrt{1-\beta t}) + C_{2i}\, Y_1(\lambda_i\sqrt{1-\beta t})\right] + \frac{U_i}{\omega_i^2}, \quad (i = 1,2,---n) \tag{4.13}$$

where $U_i$ is the amplitude of $U_i(t)$ and $C_{1i}$ and $C_{2i}$ are constants of integration which are determined by the initial conditions. Letting the initial conditions be

$$u(x,t)\Big|_{t=o} = u_o(x), \quad \frac{\partial u(x,t)}{\partial t}\Big|_{t=0} = 0 \tag{4.14}$$

and denoting

$$\int_{-L/2}^{L/2} m_o\, u_o(x)\, \varphi_i(x)\, dx = u_{oi}, \quad (i=1,2,---,n) \tag{4.15}$$

it can be shown that the solution for the axial elastic displacement is

$$u(x,t) = \sum_{i=1}^{n} \left\{ \frac{(u_{oi} - U_i/\omega_i^2)(1-\beta t)^{1/2}}{J_1(\lambda_i)\, Y_o(\lambda_i) - J_o(\lambda_i)\, Y_1(\lambda_i)} \Big[ Y_o(\lambda_i)\, J_1(\lambda_i \sqrt{1-\beta t}) \right.$$

$$\left. - J_o(\lambda_i)\, Y_1(\lambda_i \sqrt{1-\beta t}) \Big] + U_i/\omega_i^2 \right\} \varphi_i(x), \; \lambda_i = \frac{2\omega_i}{\beta} \tag{4.16}$$

Since $\lambda_i \gg 1$ we can use asymptotic expansions for Bessel functions of large argument and reduce Eq. (4.16) to

$$u(x,t) = \sum_{i=1}^{n} \left[ (u_{oi} - \frac{U_i}{\omega_i^2})(1-\beta t)^{1/4} \cos \lambda_i (1-\sqrt{1-\beta t}) + \frac{U_i}{\omega_i^2} \right] \varphi_i(x),$$

$$\lambda_i = \frac{2\omega_i}{\beta} \tag{4.17}$$

where the square bracket can be identified as $q_i(t)$.

Now we are in the position to attempt a solution for the motion $y(x,t)$. As in the case of the axial motion, we assume that the transverse elastic displacement can be written in the form of the series

$$y(x,t) = \sum_{j=1}^{n} \psi_j(x)\, \eta_j(t) \tag{4.18}$$

where $\eta_j(t)$ are generalized coordinates and $\psi_j(x)$ are eigenfunctions obtained from the solution of the eigenvalue problem defined by the differential equation

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2\psi}{dx^2}\right) = m_o\,\Omega^2\psi \tag{4.19}$$

and the boundary conditions

$$EI\,\frac{d^2\psi}{dx^2} = 0 \quad \text{and} \quad \frac{d}{dx}\left(EI\,\frac{d^2\psi}{dx^2}\right) = 0, \quad x = -L/2,\ L/2 \tag{4.20}$$

The functions $\psi_j$ are such that $\int_{-L/2}^{L/2} m_o\,\psi_i\psi_j\,dx = \delta_{ij}$, $\int_{-L/2}^{L/2} m_o\,\psi_j\,dx = 0$ and $\int_{-L/2}^{L/2} m_o\,x\,\psi_j\,dx = 0$ where the latter two expressions justify some of the simplifications in the equations for the rigid-body motion obtained in the preceding section.

Introducing Eqs. (4.3), (4.4), and (4.18) into Eq. (4.2), and recalling Eq. (4.19), we obtain

$$\sum_{j=1}^{n} m_o[(1-\beta t)\,\ddot{\eta}_j + \Omega^2_j\eta_j]\psi_j - \sum_{i=1}^{n}\ \sum_{j=1}^{n} q_i\,\frac{d}{dx}\left(EA_c\,\frac{d\varphi_i}{dx}\,\frac{d\psi_j}{dx}\right)\eta_j = 0 \tag{4.21}$$

Multiplying Eq. (4.21) by $\psi_r$ and integrating over the length of the missile, we obtain

$$(1-\beta t)\,\ddot{\eta}_r + \Omega_r^{\,2}\,\eta_r - \sum_{i=1}^{n}\ \sum_{j=1}^{n} P_{ijr}\,(t)\,\eta_j = 0 \tag{4.22}$$

where

$$P_{ijr}(t) = \int_{-L/2}^{L/2} q_i(t)\,\psi_r(x)\,\frac{d}{dx}\left(EA_c\,\frac{d\varphi_i}{dx}\ \frac{d\psi_j}{dx}\right)dx \tag{4.23}$$

In view of the boundary condition, Eqs (4.6), Eq. (4.23) reduces to

$$P_{ijr}(t) = -q_i(t)\int_{-L/2}^{L/2} EA_c\,\frac{d\varphi_i}{dx}\,\frac{d\psi_j}{dx}\,\frac{d\psi_r}{dx}\,dx \tag{4.24}$$

Unlike the set of equations for the axial motion, the set of equations for the transverse motion, Eqs. (4.22), are coupled. By retaining a limited number of terms in the expansions (4.4) and (4.18) one can uncouple Eqs. (4.22). To this end, we notice that when the integrand in Eq. (4.24) is an odd function the integral is zero. But $\varphi_i$ is odd if i is odd and $\psi_r$ is odd is r is even and vice-versa. In addition, $\varphi_i$ and $\varphi_r$ are such that if they are even functions their derivatives are odd functions and vice versa. Hence, one can write

$$
\begin{aligned}
P_{111}(t) &= -q_1(t) \int_{-L/2}^{L/2} EA_c \frac{d\varphi_1}{dx}\left(\frac{d\psi_1}{dx}\right)^2 dx = -Q_{111}\, q_1(t) \\
P_{112}(t) &= P_{121}(t) = 0 \\
P_{122}(t) &= -q_1(t) \int_{-L/2}^{L/2} EA_c \frac{d\varphi_1}{dx}\left(\frac{d\psi_2}{dx}\right)^2 dx = -Q_{122}\, q_1(t)
\end{aligned}
\tag{4.25}
$$

so that retaining only the first term in series (4.4) and the first two terms in series (4.18), Eqs. (4.22) reduce to

$$
\begin{aligned}
(1-\beta t)\,\ddot{\eta}_1 + \Omega_1^2\,\eta_1 - Q_{111}\, q_1(t)\,\eta_1 &= 0 \\
(1-\beta t)\,\ddot{\eta}_2 + \Omega_2^2\,\eta_2 - Q_{122}\, q_1(t)\,\eta_2 &= 0
\end{aligned}
\tag{4.26}
$$

which are uncoupled.

Consider the typical equation

$$
(1-\beta t)\,\ddot{\eta} + \Omega^2\eta - \left(\sum_{k=0}^{m} Q_k t^k\right)\eta = 0 \tag{4.27}
$$

and use the Frobenius method to obtain a solution. To this end we let the solution of Eq. (4.27) have the form

$$
\eta = \sum_{n=0}^{\infty} B_n t^{n+s} \tag{4.28}
$$

so that

$$\sum_{n=0}^{\infty} B_n(n + s)(n + s - 1)t^{n + s - 2} - \beta \sum_{n=0}^{\infty} B_n(n + s)\ (n + s - 1)t^{n + s - 1}$$

$$+ (\Omega^2 - Q_0) \sum_{n=0}^{\infty} B_n\, t^{n + s + 1} - Q_2 \sum_{n=0}^{\infty} B_n\, t^{n + s + 2} \text{----} = 0$$

which can be written as

$$\sum_{n=0}^{\infty} B_n(n + s)(n + s - 1)\ t^{n + s - 2} - \beta \sum_{n=1}^{\infty} \beta_{n-1}(n + s-1)(n + s - 2)t^{n+s-2}$$

$$+ (\Omega^2 - Q_0) \sum_{n=2}^{\infty} B_{n-2}\, t^{n + s - 2} - Q_1 \sum_{n=3}^{\infty} B_{n-3}\, t^{n + s - 2}$$

$$- Q_2 \sum_{n=4}^{\infty} B_{n-4}\, t^{n + s - 2} \text{----} = 0$$

The above equation is satisfied if

$$B_0\, s(s-1) = 0$$

$$B_1\, (s + 1)\, s - \beta B_0\, s(s - 1) = 0$$

$$B_2\, (s + 2)(s + 1) - \beta B_1(s + 1)\, s + B_0(\Omega^2 - Q_0) = 0 \tag{4.29}$$

$$B_3(s + 3)(s + 2) - \beta B_2(s + 2)(s + 1) + B_1(\Omega^2 - Q_0) - B_0 Q_1 = 0$$

$$B_4(s + 4)(s + 3) - \beta B_3(s + 3)(s + 2) + B_2(\Omega^2 - Q_0) - B_1 Q_1 - B_0 Q_2 = 0$$

- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

Letting a solution be given by

$$s = 0, \; B_0 \text{ and } B_1 \text{ arbitrary}$$

we can obtain from Eqs. (4.29) all of the coefficients $B_n$, $n \geq 2$, in terms of $B_0$ and $B_1$ where the latter play the role of constants of integration. For $s = 0$, Eqs. (4.29) can be written in the form of the recurrence formula

$$B_n = \beta \frac{n-2}{n} B_{n-1} - \frac{\Omega^2 - Q_0}{n(n-1)} + \frac{1}{n(n-1)} \sum_{k=1}^{\infty} Q_k B_{n-2-k} \quad ,$$

$$n \geq 2, \; B_i = 0 \text{ for } i \text{ negative} \tag{4.30}$$

Equation (4.30) yields

$$
\begin{aligned}
B_2 &= -\frac{1}{2}(\Omega^2 - Q_0)B_0 \\
B_3 &= \frac{1}{6}[-\beta(\Omega^2 - Q_0) + Q_1]B_0 - \frac{1}{6}(\Omega^2 - Q_0)B_1 \\
B_4 &= \frac{1}{12}[-\beta^2(\Omega^2 - Q_0) + \frac{1}{2}(\Omega^2 - Q_0)^2 + \beta Q_1 + Q_2]\, B_0 + \frac{1}{12}[-\beta(\Omega^2 - Q_0) + Q_1]B_1 \\
B_5 &= \frac{1}{20}[-(\beta^3 + Q_1)(\Omega^2 - Q_0) + \frac{2}{3}(\Omega^2 - Q_0)^2 + \beta^2 Q_1 + \beta Q_2 + Q_3]B_0 \\
&\quad + \frac{1}{20}[-\beta^2(\Omega^2 - Q_0) + \frac{1}{6}(\Omega^2 - Q_0)^2 + \beta Q_1 + Q_2]B_1
\end{aligned}
\tag{4.31}
$$

and upon introducing the above coefficients into Eq. (4.28) we obtain

$$
\begin{aligned}
\eta = \sum_{n=0}^{\infty} B_n t^n &= B_0 + B_1 t - \frac{1}{2}(\Omega^2 - Q_0)\, B_0 t^2 + \Big\{\frac{1}{6}[-\beta(\Omega^2 - Q_0) \\
&+ Q_1]\, B_0 - \frac{1}{6}(\Omega^2 - Q_0)\, B_1\Big\}\, t^3 + \Big\{\frac{1}{12}[-\beta^2(\Omega^2 - Q_0) + \frac{1}{2}(\Omega^2 - Q_0)^2 + \beta Q_1 \\
&+ Q_2]\, B_0 + \frac{1}{12}[-\beta(\Omega^2 - Q_0) + Q_1]\, B_1\Big\}\, t^4 + \Big\{\frac{1}{20}[-(\beta^3 + Q_1)(\Omega^2 - Q_0) \\
&+ \frac{2}{3}(\Omega^2 - Q_0)^2 + \beta^2 Q_1 + \beta Q_2 + Q_3]\, B_0 + \frac{1}{20}[-\beta^2(\Omega^2 - Q_0) + \frac{1}{6}(\Omega^2 - Q_0)^2 \\
&+ \beta Q_1 + Q_2]\, B_1\Big\}\, t^5 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
&= B_0 \Big\{ 1 - \frac{1}{2}(\Omega^2 - Q_0)t^2 + \frac{1}{6}[Q_1 - \beta(\Omega^2 - Q_0)]t^3 + \frac{1}{12}[Q_2 + \beta Q_1 \\
&+ \frac{1}{2}(\Omega^2 - Q_0)^2 - \beta^2(\Omega^2 - Q_0)]t^4 + \frac{1}{20}[Q_3 + \beta Q_2 + \beta^2 Q_1 + \frac{2}{3}(\Omega^2 - Q_0)^2 \\
&- (\beta^3 + Q_1)(\Omega^2 - Q_0)]t^5 + \text{-----------} \Big\} \qquad (4.32) \\
&+ B_1 \Big\{ t - \frac{1}{6}(\Omega^2 - Q_0)t^3 + \frac{1}{12}[Q_1 - \beta(\Omega^2 - Q_0)]t^4 + \frac{1}{20}[Q_2 + \beta Q_1 \\
&+ \frac{1}{6}(\Omega^2 - Q_0)^2 - \beta^2(\Omega^2 - Q_0 0]t^5 + \text{-----------} \Big\}
\end{aligned}
$$

Hence, one can write the solution of Eqs. (4.26) in the form

$$
\eta_j(t) = B_{0j}\,\eta_{0j}(t) + B_{1j}\,\eta_{1j}(t), \quad (j = 1,2) \qquad (4.31)
$$

where

$$
\begin{aligned}
\eta_{0j}(t) &= 1 - \frac{1}{2}(\Omega_j^2 - Q_0)t^2 + \frac{1}{6}[Q_1 - \beta(\Omega_j^2 - Q_0]t^3 + \text{---} \\
\eta_{1j}(t) &= t - \frac{1}{6}(\Omega_j^2 - Q_0)t^3 + \frac{1}{12}[Q_1 - \beta(\Omega_j^2 - Q_0)]t^4 + \text{---}
\end{aligned} \qquad (4.32)
$$

and $B_{0j}$ and $B_{1j}$, $(j = 1,2)$, are constants of integration to be determined from the initial conditions. Introducing Eq. (4.31) into Eq. (4.18) we obtain

$$
y(x,t) = \sum_{j=1}^{2} \psi_j(x) \left[ B_{0j}\,\eta_{0j}(t) + B_{1j}\,\eta_{1j}(t) \right] \qquad (4.33)
$$

Letting $y(x,0)$ and $\dot{y}(x,0)$ be the initial transverse displacement and velocity, respectively, we obtain without difficulty

$$
\begin{aligned}
B_{0j} &= \int_{-L/2}^{L/2} m_0\,\psi_j(x)\,y(x,0)\,dx \\
B_{1j} &= \int_{-L/2}^{L/2} m_0\,\psi_j(x)\,\dot{y}(x,0)\,dx
\end{aligned} \qquad (4.34)
$$

It appears that, for no external forces, $y(x,t)$ can be zero if the initial conditions are zero or they are proportional to either $\psi_j(x)$ or $x\,\psi_j(x)$, $(j \geq 2)$.

## 5. Results

The equation

$$u(x,t) = \sum_{j=1}^{n} \varphi_j(x) \left[ \left(1-\beta t\right)^{1/4} \left(u_{0j} - \frac{U_j}{\omega_j^2}\right) \cos \frac{2\omega_j}{\beta} \left(1-\sqrt{1-\beta t}\right) + \frac{U_j}{\omega_j^2} \right] \tag{5.1}$$

is not particularly amenable to numerical calculation because of the extremely large argument of the cosine term. However, for small time, t, the radical in the above term may be expanded by means of the binomial expansion formula to give

$$\cos \frac{2\omega_j}{\beta} \left[1-\left(1 - \frac{\beta t}{2} + \frac{(\beta t)^2}{8} - \ldots\right)\right] = \cos \frac{2\omega_j}{\beta}\left(\frac{\beta t}{2} - \frac{(\beta t)^2}{8} + \ldots\right) \tag{5.2}$$

so that for $t \ll 1$, we may neglect the $(\beta t)^2$ and all higher order terms in t, as well as set

$$(1 - \beta t)^{1/4} \cong 1 \tag{5.3}$$

so that a simplified expression for $u(x,t)$ appears as

$$u(x,t) \cong \sum_{j=1}^{n} \varphi_j(x) \left[ \frac{U_j}{\omega_j^2} + \left(u_{0j} - \frac{U_j}{\omega_j^2}\right) \cos \omega_j t \right] \tag{5.4}$$

Although this form has definite utility for the calculation of the axial displacement for very small t, it neglects the change in the frequency and amplitude of vibration as a function of time due to the change in the mass. However, we are interested in $u(x,t)$ for the entire period of burning since it is required for the calculation of $y(x,t)$.

With this in mind, a simple computer program has been written for the Control Data 3400 digital computer. Input to the program requires values of Young's modulus, length and area of the missile case, area of the fluid, nozzle throat area, and internal fluid pressure together with the mass per unit length and the time rate of change of the mass per unit length of the missile. The number of terms in the series is also read in as input in order to be able to evaluate the rate of convergence of the series. The output consists of values of the axial elastic displacement at discrete intervals of x for discrete values of time from time = 0 to burnout. The time increment is determined from the relation

$$\tau = \frac{h}{c} \tag{5.5}$$

where h is the spatial increment and

$$c = \sqrt{EA_c/m_c} \tag{5.6}$$

so that a sufficiently large number of deflected shapes are obtained over a cycle to ensure representative coverage. Results for 1, 2, 10, and 20 term series have been obtained for a typical solid fuel missile with the following physical properties:

$$E = 30 \cdot 10^6 \text{ lb/in}^2$$

$$A_c = 7.53 \text{ in}^2$$

$$A_F = 36.4 \text{ in}^2$$

$$A_T = 18.2 \text{ in}^2$$

$$m_0 = 0.011 \text{ lb sec}^2/\text{in}^2$$

$$\dot{m}_c = 0.004 \text{ lb sec/in}^2$$

$$L = 100 \text{ in}$$

$$\dot{p}_{L/2} = 2000 \text{ lb/in}^2$$

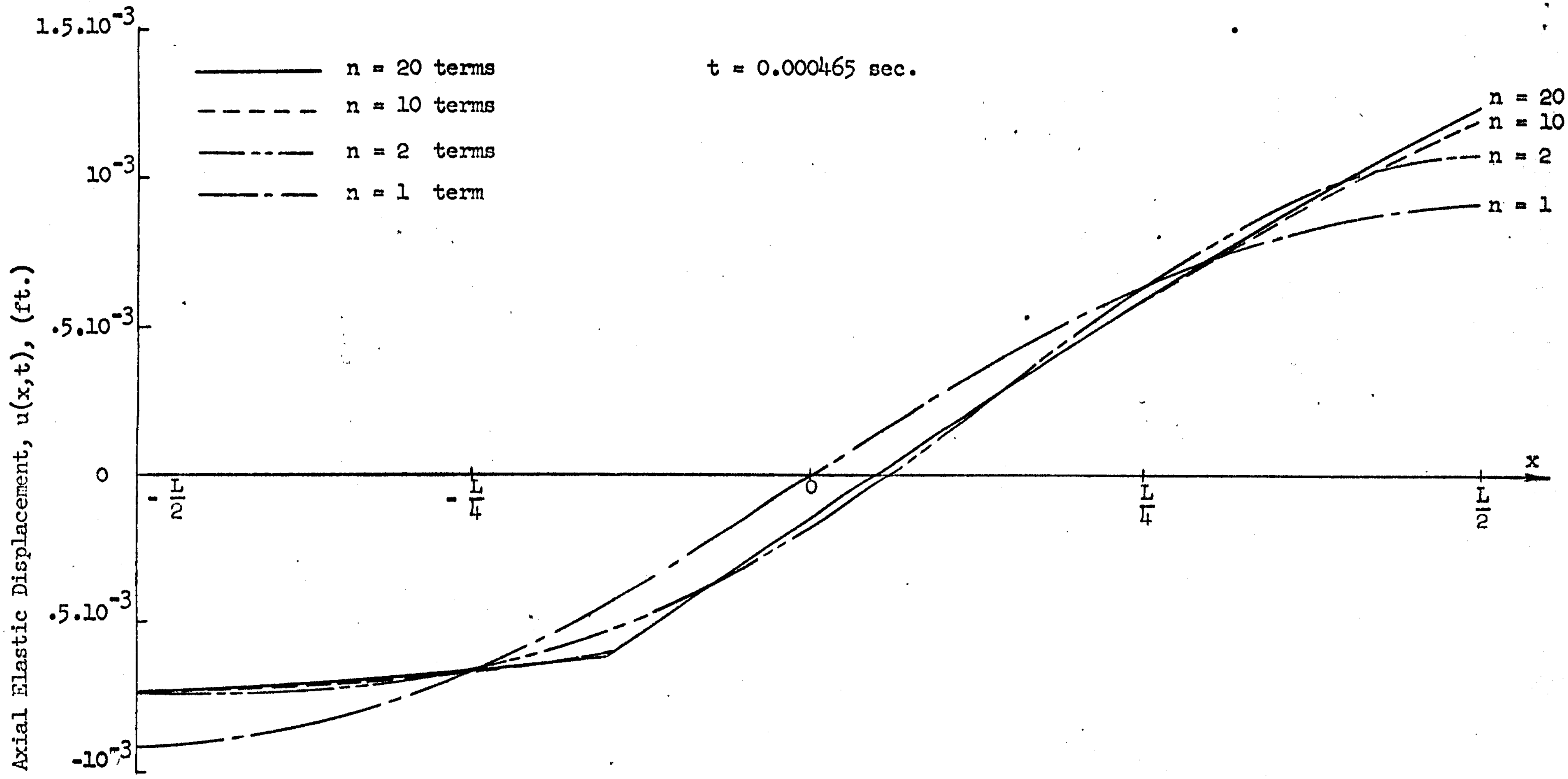
For 9 increments of length, a time increment of approximately 0.000078 sec results. A comparison of the deflected shape for different numbers of terms in the series at a typical time appears in Figure 3, while Figure 4 shows the deflected shape of the missile at several selected times through one cycle.

## 6. Conclusions and Plans for Future Work

The five coupled, nonlinear equations of motion of a solid fuel missile were derived in the first semi-annual technical report. In the present report, the problem of the internal fluid flow has been investigated in order to determine a suitable pressure and velocity distribution within the missile case. To this end, the effects of the pressure drop due to flow acceleration, the pressure increase due to the noninertial acceleration of the control-volume, and the pressure drop due to friction together with the subsequent effect on the velocity distribution were investigated. The conclusion reached was that the assumption of uniform chamber pressure and linear velocity distribution was reasonably justified.

A primary motion in the vertical direction was then postulated with the lateral rigid-body translation, the rigid-body rotation and the elastic deformations in both the axial and lateral directions considered as second order effects. A solution for the axial elastic deformation was then obtained in the form of a series of the eigenfunctions of the corresponding constant-mass systems multiplied by time-dependent generalized coordinates whose amplitudes were determined by means of the initial conditions and the forcing functions resulting from the fluid flow. It should be stressed that, although the eigenfunctions of the constant-mass missile are used, this is merely a mathematical concept and does not imply normal mode vibration. On the contrary, the system does not possess any natural frequencies in the ordinary sense, and the amplitudes are not constant at a given period in different cycles.

Unfortunately, the same series approach to the lateral vibration resulted in coupled equations due to the presence of the axial force in the equation for $y(x,t)$. In order to uncouple these equations, it was necessary



1.5.10-3
10-3
.5.10-3
0
.5.10-3
-10-3
Axial Elastic Displacement, u(x,t), (ft.)
n = 20 terms
n = 10 terms
n = 2 terms
n = 1 term
t = 0.000465 sec.
n = 20
n = 10
n = 2
n = 1
- L/2
- L/4
0
L/4
L/2
x


Figure 3

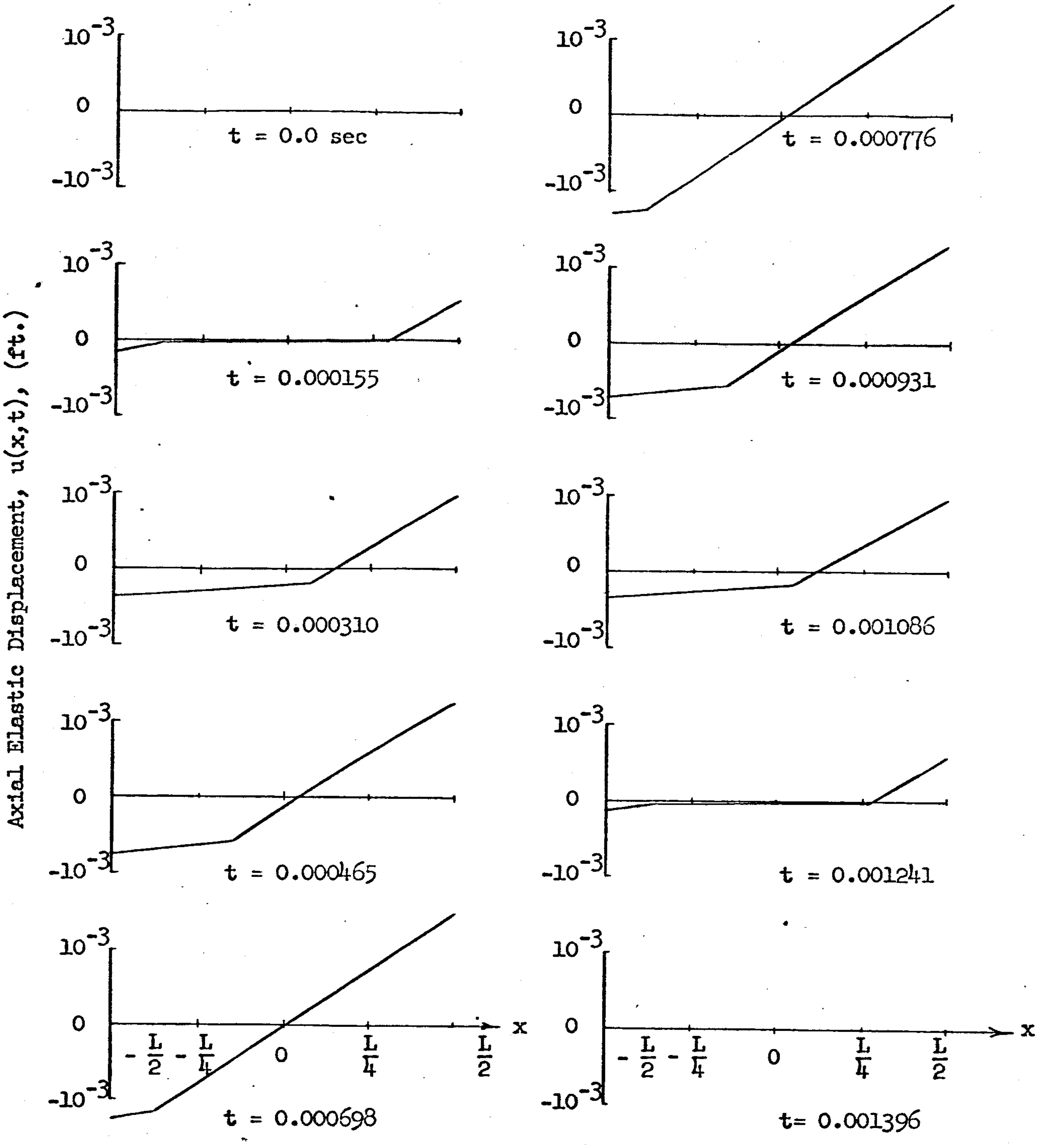

Axial Elastic Displacement, u(x,t), (ft.)
10^-3
0
-10^-3
t = 0.0 sec
t = 0.000155
t = 0.000310
t = 0.000465
t = 0.000698
t = 0.000776
t = 0.000931
t = 0.001086
t = 0.001241
t= 0.001396
- L/2 - L/4 0 L/4 L/2
x

Figure 4

to assume that the deformation in the axial direction could be approximated by the first term of the series for $u(x,t)$ and the first two terms of the series for the lateral motion $y(x,t)$. Of final consequence, it was shown that no lateral elastic motion resulted under the assumed initial conditions $y(x,0) = \dot{y}\ (x,0) = 0$.

The final phase of this project will consist of a completely numerical solution of the original five coupled, nonlinear equations of motion. Finite differences will be used together with numerical integration and an interative sequence to cope with the nonlinearity of the equations. A computer solution is obviously in order with this approach and the Control Data 3400 will be utilized. While assumptions as to the relative magnitudes of the rigid body motions as opposed to the elastic motions will not be necessary, it is envisioned that the same assumptions concerning the fluid flow will be used as are used in the present report.

## 7. References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